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A CONCENTRATION OF MASS PROPERTY FOR ISOTROPIC CONVEX BODIES IN HIGH DIMENSIONS

BY

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ABSTRACT

It is shown that, for a certain subclass of istropic convex sets in $\mathbb{R}^n$, the mass concentrates in a spherical shell, asymptotically for large n. This in turn shows that the inequality

$$1 \le \int_K |x|^2 dx \left(\int_K \frac{1}{|x|} dx \right)^2$$

is close to an equality for the mentioned class of isotropic convex sets, asymptotically for large n. It also implies a 'central limit property' for this class.

Introduction

By a **normed convex body** K in $\mathbb{R}^n$ we will understand a convex compact set of volume 1 whose center of inertia is at 0. The normed convex body is **isotropic** if its ellipsoid of inertia is a Euclidean ball. The set of all isotropic normed convex bodies in $\mathbb{R}^n$ will be denoted by $\mathcal{K}_n$.

For $K \in \mathcal{K}_n$, $u \in S_{n-1}$ (unit sphere in $\mathbb{R}^n$) we define

$$\varphi_{K,u}(t) := \lambda_{n-1}(\{x \in K; x \cdot u = t\}) \quad (t \in \mathbb{R}).$$

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In [5] it was shown that for several examples $\varphi_{K,u}$ is close to a Gaussian density for large n and certain directions u. As one of the few general results we noted that, for

$$\varphi_K(0) := \int_{S_{n-1}} \varphi_{K,u}(0) d\mu_{n-1}(u)$$

(μ_{n-1} the normed surface measure on S_{n-1}), one obtains

$$\liminf_{n\to\infty} \inf_{K\in\mathcal{K}_n} \frac{\varphi_K(0)}{g_{L_K^2}(0)} \geq 1, \tag{0.1}$$

where $g_{L_K^2}$ is the Gaussian density corresponding to K (cf. [5; Prop. 1.3]). The source for this result is the inequality

$$1 \leq \int_K |x|^2 dx \left(\int_K \frac{1}{|x|} dx \right)^2 \tag{0.2}$$

(Hölder's inequality). It is one of the aims of this note to show that, for a certain subclass of $\mathcal{K}_n$, the inequality (0.2) is close to an equality, for large n. This implies that for this subclass the converse inequality in (0.1) (formulated suitably) is valid.

For $K \in \mathcal{K}_n$ we denote by L_K the radius of the ball of inertia of K,

$$L_K^2 := \int_K (x \cdot u)^2 dx,$$

independently of $u \in S_{n-1}$ (following the notation of [6]). The source of the result formulated above is the observation that, for the subclass of $\mathcal{K}_n$ mentioned above, the mass of K is concentrated in a neighbourhood of a sphere with radius $\sqrt{n}L_K$, for $n \to \infty$.

The method to obtain these results is to prove certain estimates ((1.2) and (1.3) in Section 1) for the second and fourth moments of the sets in our class. The class satisfying these estimates contains the normed Euclidean balls, cubes, cross polytopes, and regular simplices.

It seems worthwhile mentioning a connection between the phenomenon described here and the well-studied concentration of measure property; cf. [7], [8]. It is known that one has an 'inverse Hölder inequality'

$$\left(\int_K |x|^4 dx \right)^{\frac{1}{4}} \leq c \left(\int_K |x|^2 dx \right)^{\frac{1}{2}},$$

with some constant c independent of $K \in \mathcal{K}_n$ and $n \in \mathbb{N}$; cf. [6; Sec. 1.4], [3], [1]. We show for our subclass that c can be chosen close to 1 for large n.

In Section 1 we show that a suitable estimate on the fourth moment gives control of the concentration of mass described above, and we establish the connection to the asymptotic equality in (0.2). Moreover, we recall and explain that our results imply certain 'central limit properties'.

In the remaining sections we show that the class satisfying (1.2), (1.3) has certain saturation properties. In Section 2 we show that forming cones does not lead out of the class. In Sections 3 and 4 it is shown that the class is also stable under forming cartesian products and 'joins', respectively.

In Section 6 we indicate some explicit expressions for cubes and cross polytopes.

We note that a large part of the paper is purely calculational. We have tried to always indicate major steps, so the reader should have sufficiently many hints to carry out the additional computations.

The author acknowledges useful conversations with U. Brehm and H. Vogt on the topics of this paper. In particular, U. Brehm initiated and contributed some of the computations of Sections 3 and 4.

1. Preliminaries

In this section we show how the asymptotic equality in (0.2) follows from a concentration of mass phenomenon, and how this latter property can be derived from estimates of moments.

1.1 Proposition: *Let $\mathcal{T} \subseteq \bigcup_{n\in\mathbb{N}} \mathcal{K}_n$, and assume*

(i) $\sup_{K\in\mathcal{T}} L_K < \infty$,

(ii) $\sup_{K\in\mathcal{K}_n\cap\mathcal{T}} \lambda_n \left(\{x \in K;\ \left||x|^2 - nL_K^2\right| \geq \varepsilon n L_K^2\}\right) \longrightarrow 0 \quad (n \to \infty)$, *for all* $\varepsilon > 0$.

Then

$$\sup_{K\in\mathcal{K}_n\cap\mathcal{T}} \int_K |x|^2 dx \left(\int_K \frac{1}{|x|} dx\right)^2 \longrightarrow 1 \quad (n \to \infty).$$

1.2 Remarks: (a) In [6; Sec. 5], the validity of assumption (i) for $\mathcal{T} = \bigcup_{n\in\mathbb{N}} \mathcal{K}_n$ is stated as 'probably a generally accepted hypothesis', and several equivalent formulations are discussed.

(b) U. Brehm [4] communicated that, under the assumptions of Proposition 1.1, one can show that the averaged φ_K (cf. [5; Sec. 1]) is close to the corresponding Gaussian density, for $K \in \mathcal{T}$ and large n.

(c) Combining the hypotheses of Proposition 1.1 with a result of von Weizsäcker [9] one obtains the following conclusion.

Let κ be a metric which induces convergence in law on the set of probability measures on $\mathbb{R}$, and assume that $\mathcal{T}$ has the properties assumed in Proposition 1.1.

Let $\varepsilon > 0$. Then

$$\sup_{K\in\mathcal{K}_n\cap\mathcal{T}} \mu_{n-1}(\{u \in S_{n-1};\, \kappa(\varphi_{K,u}\lambda_1, g_{L_K^2}\lambda_1) > \varepsilon\}) \longrightarrow 0 \qquad (n \to \infty),$$

where λ_1 is the 1-dimensional Lebesgue measure on $\mathbb{R}$, and

$$g_{\sigma^2}(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-t^2/2\sigma^2}.$$

This statement follows from [9; Corollary 1]; note that condition (ii) of Proposition 1.1 implies that '$\int \mathcal{N}(|x|^2/n)P'(dx)$' (cf. [9]) $= \int_K g_{|x|^2/n} dx \lambda_1$ is close to $g_{L_K^2}\lambda_1$ for large n.

The statement given above is a 'central limit property' of the set $\mathcal{T}$; cf. [5; Definition 1.1].

On the one hand, this statement is stronger than the fact stated in (b) since it gives a conclusion for the individual marginal measures; on the other hand, the approximation of the measures is obtained only in a weaker sense.

Before starting the proof of Proposition 1.1 we recall Stirling's formula: For $x > 0$ one has

$$\left(\frac{x}{e}\right)^x \sqrt{2\pi x} \le \Gamma(x+1) \le \left(\frac{x}{e}\right)^x \sqrt{2\pi x}\, e^{\vartheta(x)/12x},$$

where $0 \le \vartheta(x) \le 1$. We refer to [5; end of Sec. 1] for precise references.

Proof of Proposition 1.1: Let $0 < \varepsilon < 1$. Choose $0 < c \le 1/\sqrt{2\pi e}$. Then

$$\sqrt{n}\int_K \frac{1}{|x|}dx = \sqrt{n}\int_{\substack{K\\ |x|\le c\sqrt{n}}} \frac{1}{|x|}dx + \sqrt{n}\int_{\substack{K\\ c\sqrt{n}\le|x|\le\sqrt{(1-\varepsilon)n}L_K}} \frac{1}{|x|}dx + \sqrt{n}\int_{\substack{K\\ \sqrt{(1-\varepsilon)n}L_K\le|x|}} \frac{1}{|x|}dx.$$

The first term is estimated from above by

$$\begin{aligned}\sqrt{n}\int_{|x|\le c\sqrt{n}} \frac{1}{|x|}dx &= \frac{n}{n-1}\frac{\pi^{n/2}}{\Gamma(n/2+1)} c^{n-1} n^{n/2} \\ &\le \frac{n}{n-1}\frac{\pi^{n/2}}{\left(\frac{n}{2e}\right)^{n/2}\sqrt{2\pi\frac{n}{2}}} c^{n-1} n^{n/2} \\ &= \frac{n}{n-1}\left(\sqrt{2\pi e}c\right)^n \frac{1}{c\sqrt{\pi n}} \longrightarrow 0.\end{aligned}$$

The second term is estimated by

$$\sqrt{n}\frac{1}{c\sqrt{n}}\int_{\substack{K\\ |x|\le\sqrt{(1-\varepsilon)n}L_K}} dx \le \frac{1}{c}\sup_{K\in\mathcal{K}_n\cap\mathcal{T}} \lambda_n\left(\{x\in K;\, \left||x|^2 - nL_K^2\right| \ge \varepsilon n L_K^2\}\right) \longrightarrow 0.$$

The third term is estimated by

$$\sqrt{n}\frac{1}{\sqrt{(1-\varepsilon)n}L_K} = \frac{1}{\sqrt{1-\varepsilon}L_K}.$$

As a consequence we obtain

$$\limsup_{n\to\infty} \sup_{K\in\mathcal{K}_n\cap\mathcal{T}} \frac{1}{n}\int_K |x|^2 dx \left(\sqrt{n}\int_K \frac{1}{|x|}dx\right)^2 \leq \frac{1}{1-\varepsilon}.$$

(Note that it is at this point that we use assumption (i); observe $\int_K |x|^2 dx = nL_K^2$.) Since this holds for all $0<\varepsilon<1$ one obtains the assertion. ∎

In order to derive assumption (ii) of Proposition 1.1 from estimates on moments, we use Chebychev's inequality. For isotropic convex bodies K in $\mathbb{R}^n$ (not necessarily normed to volume 1) we define

$$M_{0,n}(K) := \lambda_n(K),$$
$$M_{2,n}(K) := \int_K |x|^2 dx,$$
$$M_{4,n}(K) := \int_K |x|^4 dx.$$

Then, for $K \in \mathcal{K}_n$, one has

$$\begin{aligned} &\lambda_n\left(\{x\in K;\ \left||x|^2 - nL_K^2\right| \geq \varepsilon n L_K^2\}\right) \\ &\leq \frac{1}{\varepsilon^2 n^2 L_K^4}\int_K \left||x|^2 - nL_K^2\right|^2 dx \\ &= \frac{1}{\varepsilon^2 n^2 L_K^4}\left(\int_K |x|^4 dx - 2nL_K^2\int_K |x|^2 dx + n^2 L_K^4\right) \\ &= \frac{1}{\varepsilon^2 n^2 L_K^4}\left(M_{4,n}(K) - n^2 L_K^4\right) \\ &= \frac{1}{\varepsilon^2}\left(\frac{M_{4,n}(K)}{M_{2,n}(K)^2} - 1\right). \end{aligned} \tag{1.1}$$

If K is isotropic convex, not necessarily normed to volume one, then $M_{4,n}(K)/M_{2,n}(K)^2$ has to be replaced by

$$\frac{M_{4,n}}{M_{0,n}^{n+4/n}}\cdot\frac{M_{0,n}^{2n+4/n}}{M_{2,n}^2} = \frac{M_{4,n}\cdot M_{0,n}}{M_{2,n}^2},$$

where the beginning of the estimate (1.1) refers to the associated normed isotropic convex body.

We are going to consider the class $\mathcal{T}$ satisfying the inequalities

$$M_{2,n}(K) \le \frac{n}{(n+1)(n+2)}\frac{n!^{2/n}}{(n+1)^{1/n}}, \tag{1.2}$$

$$\frac{M_{4,n}(K)}{M_{2,n}(K)^2} \le \frac{(n+1)(n+2)}{(n+3)(n+4)}\left(1+\frac{2}{n}+\frac{6}{n+1}\right). \tag{1.3}$$

It will turn out that in both of these inequalities, one has equality for the n-dimensional normed regular simplex Δ_n.

Let us note immediately that (1.2) implies (i) of Proposition 1.1, by Stirling's formula:

$$L_K^2 = \frac{1}{n}M_{2,n}(K) \le \frac{n!^{2/n}}{(n+1)^{1+1/n}(n+2)} \approx \frac{\left(\frac{n}{e}\right)^2(2\pi n)^{1/n}}{(n+1)^{1+1/n}(n+2)} \longrightarrow \frac{1}{e^2}.$$

Further, estimate (1.3) together with (1.1) implies that assumption (ii) of Proposition 1.1 is satisfied.

More precisely, we define

$$\mathcal{T} := \bigcup_{n\in\mathbb{N}} \mathcal{T}_n, \tag{1.4}$$

with

$$\mathcal{T}_n := \{K \in \mathcal{K}_n;\ K \text{ satisfies (1.2), (1.3)}\} \quad (n \in \mathbb{N}), \tag{1.5}$$

and we show that $\mathcal{T}$ contains certain types of sets and is saturated with respect to certain operations.

To begin with we mention that $\mathcal{T}$ contains all (normed) Euclidean balls. In fact, for a ball of radius 1, one has (with the volume

$$\tau_n = \frac{\pi^{n/2}}{\Gamma(n/2+1)}$$

of the unit ball in $\mathbb{R}^n$ and the $(n-1)$-dimensional volume

$$\sigma_{n-1} = n\tau_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

of S_{n-1})

$$M_{0,n} = \tau_n = \frac{\sigma_{n-1}}{n},$$

$$M_{2,n} = \int |x|^2 dx = \sigma_{n-1}\int_0^1 r^{n+1}dx = \frac{\sigma_{n-1}}{n+2},$$

$$M_{4,n} = \int |x|^4 dx = \frac{\sigma_{n-1}}{n+4}.$$

Thus,

$$\frac{M_{4,n}M_{0,n}}{M_{2,n}^2} = \frac{(n+2)^2}{(n+4)n}$$

satisfies (1.3) (by a simple comparison). Moreover, (1.2) holds as soon as one knows that the right hand side of (1.2) is $M_{2,n}$ for the simplex (see below), since clearly $M_{2,n}$ is smallest for the ball in $\mathcal{K}_n$.

Note also that for $n = 1$, i.e., $K = [-\frac{1}{2}, \frac{1}{2}]$, one has equality in (1.2) and (1.3).

We summarize the results of the subsequent sections as well as the preceding discussion in the following statement.

1.3 THEOREM: *The set $\mathcal{T}$ defined in (1.4), (1.5) is saturated with respect to forming cones, cartesian products and joins, and $\mathcal{T}$ contains the normed Euclidean balls, cubes, cross polytopes and regular simplices.*

The set $\mathcal{T}$ satisfies the hypotheses of Proposition 1.1, and therefore $\mathcal{T}$ has the 'central limit property' formulated in Remark 1.2 (c).

1.2 Remark: In [2] — this preprint came to the author's knowledge only after submitting the present paper — a central limit property is proved for a subset of $\bigcup_{n\in\mathbb{N}} \mathcal{K}_n$. The key point of this paper is a 'concentration hypothesis' corresponding to our property (ii) in Proposition 1.1. Using (1.3) together with (1.1) it is elementary to show

$$\lambda_n\left(\left\{x \in K;\ \left|\frac{|x|^2}{n} - L_K^2\right| \geq r\right\}\right) \leq \frac{4L_K^4}{nr^2} \tag{1.6}$$

for all $r > 0$, $n \in \mathbb{N}$, $K \in \mathcal{T} \cap \mathcal{K}_n$. This inequality — with the constant 35 instead of 4 — is proved in [2; Theorem 5] for l_p^n balls. Inequality (1.6) implies the 'concentration hypothesis' and therefore the central limit property of [2]. The main differences to the statement in Remark 1.2 (c) are: 1. The treatment in [2] is restricted to symmetric bodies. 2. The closeness of the marginal densities to the Gaussian density is expressed in terms of the distribution of the measures. 3. The results of [2] are quantitative, including asymptotic estimates.

2. Cones over isotropic convex bodies; the regular simplex

Let $K \subseteq \mathbb{R}^{n-1}$ be convex, isotropic. We denote by K_h the cone over K with height h whose center of gravity is at 0. This means that its basis is the set

$$\left\{x \in \mathbb{R}^n;\ (x_1, \ldots, x_{n-1}) \in K,\ x_n = -\frac{1}{n+1}h\right\},$$

and its apex is the point $\left(0,\dots,0,\frac{n}{n+1}h\right)$. The weight function determining the size of K_h at $x_n = t$ is then

$$-\frac{1}{h}\left(t-\frac{n}{n+1}h\right)=\frac{n}{n+1}-\frac{t}{h}.$$

One obtains

$$M_{0,n}(K_h)=\frac{h}{n}M_{0,n-1}(K). \tag{2.1}$$

In order to obtain $M_{2,n}(K_h)$ we calculate

$$\int\limits_{K_h} x_n^2dx=\dots=\left(\frac{n}{n+1}\right)^{n+2}h^3\int\limits_{-1/n}^{1}s^2(1-s)^{n-1}dsM_{0,n-1}(K)$$

$$\Bigg[\int\limits_{-1/n}^{1} s^2(1-s)^k ds=\cdots$$

$$=\left(\frac{n+1}{n}\right)^{k+1}\left(\frac{2}{(k+1)(k+2)(k+3)}-\frac{2}{n(k+2)(k+3)}+\frac{1}{n^2(k+3)}\right);$$

$$\text{for } k=n-1\text{:}\quad =\dots=\left(\frac{n+1}{n}\right)^{n}\frac{1}{n^2(n+2)}\Bigg]$$

$$=\left(\frac{n}{n+1}\right)^{n+2}h^3\frac{(n+1)^n}{n^{n+2}(n+2)}M_{0,n-1}(K)$$

$$=\frac{h^3}{(n+1)^2(n+2)}M_{0,n-1}(K)$$

and

$$\int_{K_h}(x_1^2+\dots x_{n-1}^2)dx=\dots=\frac{h}{n+2}M_{2,n-1}(K). \tag{2.2}$$

For isotropy of K_h one needs to have

$$\frac{1}{n-1}\int_{K_h}(x_1^2+\dots+x_{n-1}^2)dx=\int_{K_h}x_n^2dx,$$

which amounts to

$$h^2=\frac{(n+1)^2}{n-1}\frac{M_{2,n-1}(K)}{M_{0,n-1}(K)}. \tag{2.3}$$

From (2.2) one obtains, assuming $M_{0,n-1}(K)=1$,

$$M_{2,n}(K_h)=\frac{n}{n-1}\int_{K_h}(x_1^2+\dots+x_{n-1}^2)dx=\frac{n}{(n-1)(n+2)}hM_{2,n-1}(K). \tag{2.4}$$

For the resulting normed convex body $\hat{K}_h$ we therefore obtain, taking into account (2.3),

$$M_{2,n}(\hat{K}_h) = \frac{M_{2,n}(K_h)}{M_{0,n}(K_h)^{(n+2)/n}}$$
$$= \frac{n^{2(n+1)/n}}{(n+2)(n-1)^{1-1/n}(n+1)^{2/n}} M_{2,n-1}(K)^{(n-1)/n}.$$

Assuming (1.2) for K (with $n-1$) we thus obtain (1.2) for $\hat{K}_h$ (short computation). Moreover, if (1.2) is satisfied with equality for $K = \Delta_{n-1}$ then (1.2) is satisfied with equality for $\Delta_n = \hat{K}_h$; hence induction shows equality in (1.2) for the n-dimensional regular simplex.

In order to show the persistence of (1.3) we calculate

$$M_{4,n}(K_h) = \int_{K_h} |x|^4 dx = \int_{-h/(n+1)}^{nh/(n+1)} \int_{(n/(n+1)-t/h)K} \left(|y|^2 + t^2\right)^2 dydt$$
$$= \iint |y|^4 dydt + 2\iint |y|^2 t^2 dydt + \iint t^4 dydt$$
$$= \mathrm{I} + \mathrm{II} + \mathrm{III},$$

$$\mathrm{I} = \int_{-h/(n+1)}^{nh/(n+1)} \left(\frac{n}{n+1} - \frac{t}{h}\right)^{n+3} dt\, M_{4,n-1}(K) = \cdots = \frac{h}{n+4} M_{4,n-1}(K),$$

$$\mathrm{II} = 2\int_{-h/(n+1)}^{nh/(n+1)} t^2 \left(\frac{n}{n+1} - \frac{t}{h}\right)^{n+1} dt\, M_{2,n-1}(K)$$
$$= 2\left(\frac{n}{n+1}\right)^{n+4} h^3 \int_{-1/n}^{1} s^2(1-s)^{n+1} ds\, M_{2,n-1}(K)$$

[for the integral apply the formula derived above, with $k = n+1$;

$\int_{-1/n}^{1} s^2(1-s)^{n+1} ds = \cdots = \frac{(n+1)^{n+2}(n^2+n+6)}{n^{n+4}(n+2)(n+3)(n+4)}$]

$$= 2\tfrac{n^2+n+6}{(n+1)^2(n+2)(n+3)(n+4)} h^3\, M_{2,n-1}(K),$$

$$\mathrm{III} = \int_{-h/(n+1)}^{nh/(n+1)} t^4 \left(\frac{n}{n+1} - \frac{t}{h}\right)^{n-1} dt\, M_{0,n-1}(K)$$

$$= \left(\frac{n}{n+1}\right)^{n+4} h^5 \int_{-1/n}^{1} s^4(1-s)^{n-1} ds\, M_{0,n-1}(K)$$

$$\Big[\int_{-1/n}^{1} s^4(1-s)^k ds = \cdots$$

$$= \left(\frac{n+1}{n}\right)^{k+1} \left(\frac{1}{k+1} - \frac{4}{k+2} + \frac{6}{k+3} - \frac{4}{k+4} + \frac{1}{k+5} - \frac{4}{n}\left(\frac{1}{k+2} - \frac{3}{k+3} + \frac{3}{k+4} - \frac{1}{k+5}\right) + \frac{6}{n^2}\left(\frac{1}{k+3} - \frac{2}{k+4} + \frac{1}{k+5}\right) - \frac{4}{n^3}\left(\frac{1}{k+4} - \frac{1}{k+5}\right) + \frac{1}{n^4}\frac{1}{k+5}\right)$$

$$= \left(\frac{n+1}{n}\right)^{k+1} \left(\frac{4!}{(k+1)\cdots(k+5)} - \frac{4\cdot 3!}{n(k+2)\cdots(k+5)} + \frac{6\cdot 2!}{n^2(k+3)\cdots(k+5)} - \frac{4}{n^3(k+4)(k+5)} + \frac{1}{n^4(k+5)}\right);$$

$$\text{for } k = n-1: \quad = \cdots = \frac{(n+1)^n}{n^{n+4}} \cdot 3 \cdot \frac{3n^2 - n + 2}{(n+2)(n+3)(n+4)}\Big]$$

$$= \frac{3(3n^2 - n + 2)}{(n+1)^4(n+2)(n+3)(n+4)} h^5 M_{0,n-1}(K).$$

Taking into account (2.3) one obtains (abbreviating $M_{j,n-1} := M_{j,n-1}(K)$, $M_{j,n} := M_{j,n}(K_h)$)

$$\begin{aligned} M_{4,n} =& \frac{h}{n+4} M_{4,n-1} + 2h \frac{n^2+n+6}{(n-1)(n+2)(n+3)(n+4)} \frac{M_{2,n-1}^2}{M_{0,n-1}} \\ &+ 3h \frac{3n^2 - n + 2}{(n-1)^2(n+2)(n+3)(n+4)} \frac{M_{2,n-1}^2}{M_{0,n-1}} \\ =& \frac{h}{n+4} M_{4,n-1} + h \frac{2n-1}{(n-1)^2(n+4)} \frac{M_{2,n-1}^2}{M_{0,n-1}}, \end{aligned}$$

and, using (2.1) and (2.4),

$$\frac{M_{4,n}M_{0,n}}{M_{2,n}^2} = \frac{(n-1)^2(n+2)^2}{n^3(n+4)}\frac{M_{4,n-1}M_{0,n-1}}{M_{2,n-1}^2} + \frac{(n+2)^2(2n-1)}{n^3(n+4)}. \tag{2.5}$$

Now, assuming (1.3) for K (with $n-1$), a straightforward computation shows (1.3) for K_h, with equality if equality holds for K. The latter shows that (1.3) holds with equality for the simplex Δ_n. In fact, formula (2.5) is the source for the expression on the right hand side of (1.3). (Thanks to H. Vogt!)

Summarizing the previous discussion we have shown that $\mathcal{T}$ contains all the normed regular simplices Δ_n, with equalities in (1.2) and (1.3), and that $\mathcal{T}$ is invariant under formation of cones $\hat{K}_h$ with a basis $K \in \mathcal{T}$.

3. Cartesian products

Let $K_j \in \mathcal{K}_{n_j}$ for $j = 1, 2$, $n := n_1 + n_2$. Then

$$K := \left(\lambda^{1/n_1}K_1\right) \times \left(\lambda^{-1/n_2}K_1\right) \in \mathcal{K}_n$$

for

$$\lambda = \left(\frac{\frac{1}{n_2}M_{2,n_2}(K_2)}{\frac{1}{n_1}M_{2,n_1}(K_1)}\right)^{\frac{n_1 n_2}{2n}}, \tag{3.1}$$

and

$$\frac{1}{n}M_{2,n}(K) = \left(\frac{1}{n_1}M_{2,n_1}(K_1)\right)^{\frac{n_1}{n}}\left(\frac{1}{n_2}M_{2,n_2}(K_2)\right)^{\frac{n_2}{n}}. \tag{3.2}$$

Assuming (1.2) for K_1, K_2, the desire to show (1.2) for K amounts to showing

$$\frac{(n_1!n_2!)^{2/n}}{(n_1+1)^{(n_1+1)/n}(n_1+2)^{n_1/n}(n_2+1)^{(n_2+1)/n}(n_2+2)^{n_2/n}} \le \frac{n!^{2/n}}{(n+1)^{(n+1)/n}(n+2)},$$

or

$$\frac{(n_1!n_2!)^2}{(n_1+1)^{n_1+1}(n_1+2)^{n_1}(n_2+1)^{n_2+1}(n_2+2)^{n_2}} \le \frac{n!^2}{(n+1)^{n+1}(n+2)^n}. \tag{3.3}$$

This is true for $n_1 = 0$, $n_2 = n$, with equality. Let $0 \le n_1 < n_2$. It is sufficient to show that the left hand side of (3.3) decreases if (n_1, n_2) is replaced by

(n_1+1, n_2-1), i.e.,

$$\frac{(n_1!n_2!)^2}{(n_1+1)^{n_1+1}(n_1+2)^{n_1}(n_2+1)^{n_2+1}(n_2+2)^{n_2}} \geq \frac{(n_1+1)!^2(n_2-1)!^2}{(n_1+2)^{n_1+2}(n_1+3)^{n_1+1}n_2^{n_2}(n_2+1)^{n_2-1}},$$

or

$$\frac{n_2^{n_2+2}}{(n_2+1)^2(n_2+2)^{n_2}} \geq \frac{(n_1+1)^{n_1+3}}{(n_1+2)^2(n_1+3)^{n_1+1}}.$$

This means that the assertion is reduced to showing that

$$\frac{n^{n+2}}{(n+1)^2(n+2)^n}$$

is increasing for $n \geq 1$. This is calculated directly for $n = 1, 2$, and it is now sufficient to show that

$$f(x) = \frac{x^{x+2}}{(x+1)^2(x+2)^x}$$

is increasing for $x \geq 2$. Now

$$\begin{aligned} \frac{d}{dx}(\ln f(x)) &= -\ln\left(1+\frac{2}{x}\right) + \frac{2}{x} - \frac{2}{x+1} + \frac{2}{x+2} \\ &= -\left(\frac{2}{x} - \frac{1}{2}\left(\frac{2}{x}\right)^2 + \frac{1}{3}\left(\frac{2}{x}\right)^3 - \frac{1}{4}\left(\frac{2}{x}\right)^4 + \cdots\right) \\ &\quad + \frac{2}{x} - \frac{2}{x+1} + \frac{2}{x+2}. \end{aligned}$$

The sum of the first three terms of the series of the logarithm and the terms outside the series is

$$\frac{2(x-2)(5x+4)}{3x^3(x+1)(x+2)} \geq 0$$

for $x \geq 2$. Also, the remaining part of the series is ≥ 0 (alternating signs, and decreasing absolute values).

In order to discuss (1.3) we calculate

$$\begin{aligned} M_{4,n}(K) &= \int_K \left(|x'|^2 + |x''|^2\right)^2 dx \\ &= \int_K |x'|^4 dx + 2\int_K |x'|^2|x''|^2 dx + \int_K |x''|^4 dx \\ &= \lambda^{(n_1+4)/n_1} M_{4,n_1}(K_1)\lambda^{-1} \\ &\quad + 2\lambda^{(n_1+2)/n_1} M_{2,n_1}(K_1)\lambda^{-(n_2+2)/n_2} M_{2,n_2}(K_2) \\ &\quad + \lambda\lambda^{-(n_2+4)/n_2} M_{4,n_2}(K_2) \end{aligned}$$

(inserting (3.1), and using an obvious simplified notation)

$$= \left(\frac{1}{n_1}M_{2,n_1}\right)^{2\frac{n_1}{n}} \left(\frac{1}{n_2}M_{2,n_2}\right)^{2\frac{n_2}{n}}$$
$$\times \left(\left(\frac{1}{n_1}M_{2,n_1}\right)^{-2} M_{4,n_1} + 2n_1n_2 + \left(\frac{1}{n_2}M_{2,n_2}\right)^{-2} M_{4,n_2} \right).$$

Using (3.2) we finally obtain

$$\frac{M_{4,n}}{M_{2,n}^2} = \frac{1}{n^2}\left(n_1^2 \frac{M_{4,n_1}}{M_{2,n_1}^2} + 2n_1n_2 + n_2^2 \frac{M_{4,n_2}}{M_{2,n_2}^2} \right).$$

Assuming (1.3) for K_1, K_2, the desire to show (1.3) for K then means showing the inequality

$$g(n_1) + 2n_1n_2 + g(n_2) \le g(n) \tag{3.4}$$

with

$$\begin{aligned} g(k) &= k^2 \frac{(k+1)(k+2)}{(k+3)(k+4)} \left(1 + \frac{2}{k} + \frac{6}{k+1}\right) \\ &= k^2 + 4k - 4\left(\frac{5k^2 + 11k}{(k+3)(k+4)}\right) \\ &= k^2 + 4k - 4h(k). \end{aligned}$$

Taking into account $n_1^2 + 2n_1n_2 + n_2^2 = n^2$, $4n_1 + 4n_2 = 4n$, (3.4) is transformed to

$$h(n_1) + h(n_2) \ge h(n). \tag{3.5}$$

For $n_1 = 0$, $n_2 = n$ one has equality in (3.5). It is now sufficient to show: If $0 \le n_1 < n_2$ and one replaces (n_1, n_2) in the left hand side of (3.5) by (n_1+1, n_2-1), then this expression increases, i.e.,

$$h(n_1) + h(n_2) \le h(n_1 + 1) + h(n_2 - 1),$$

or

$$h(n_2) - h(n_2 - 1) \le h(n_1 + 1) - h(n_1),$$

which means that $k \mapsto h(k+1) - h(k)$ is decreasing for $k \ge 0$. The latter follows from

$$h(k+1) - h(k) = 24\frac{k+2}{(k+3)(k+4)(k+5)}.$$

Summarizing we have shown that $\mathcal{T}$ is invariant under formation of isotropic cartesian products of sets in $\mathcal{T}$. In particular, $\mathcal{T}$ contains all cubes $\left[-\frac{1}{2}, \frac{1}{2}\right]^n$.

4. The join

Let $K_j \in \mathcal{K}_{n_j}$ for $j = 1, 2$, $n := n_1 + n_2$. We form the **join** $K_1 * K_2 \subseteq \mathbb{R}^n$,

$$K_1 * K_2 := \{(tx', (1-t)x''); \, x' \in K_1, \, x'' \in K_2, \, 0 \le t \le 1\}.$$

Then

$$\lambda_n(K_1 * K_2) = \int_0^1 (1-t)^{n_2} \lambda_{n_2}(K_2) n_1 t^{n_1 - 1} \lambda_{n_1}(K_1) dt$$

(*Idea*: The differential volume $n_1 t^{n_1-1} \lambda_{n_1}(K_1) dt$ of the one-parameter family $t \mapsto tK_1$ is multiplied by the volume of $(1-t)K_2$.)

$$= n_1 \int_0^1 t^{n_1-1}(1-t)^{n_2} dt = n_1 \mathrm{B}(n_1, n_2 + 1)$$
$$= n_1 \frac{\Gamma(n_1)\Gamma(n_2+1)}{\Gamma(n_1+n_2+1)} = \frac{n_1! n_2!}{n!}.$$

Next we calculate the second moments of $K_1 * K_2$:

$$\int_{K_1 * K_2} |x''|^2 dx = n_1 \int_0^1 t^{n_1-1} \lambda_{n_1}(K_1) \int_{(1-t)K_2} |x''|^2 dx_2 dt$$
$$= \frac{n_1!(n_2+2)!}{(n+2)!} M_{2,n_2},$$
$$\int_{K_1 * K_2} |x'|^2 dx = \frac{(n_1+2)! n_2!}{(n+2)!} M_{2,n_1}.$$

Isotropy requires

$$\frac{1}{n_2} \int_{\alpha_1 K_1 * \alpha_2 K_2} |x''|^2 dx = \frac{1}{n_1} \int_{\alpha_1 K_1 * \alpha_2 K_2} |x'|^2 dx,$$
$$\frac{1}{n_2} \alpha_1^{n_1} \alpha_2^{n_2+2} \frac{n_1!(n_2+2)!}{(n+2)!} M_{2,n_2} = \frac{1}{n_1} \alpha_1^{n_1+2} \alpha_2^{n_2} \frac{(n_1+2)! n_2!}{(n+2)!} M_{2,n_1},$$

$$\alpha_2^2 n_1 (n_2+1)(n_2+2) M_{2,n_2} = \alpha_1^2 n_2 (n_1+1)(n_1+2) M_{2,n_1}. \tag{4.1}$$

Norming $\alpha_1 K_1 * \alpha_2 K_2$ to volume 1 yields

$$\alpha_1^{n_1} \alpha_2^{n_2} = \frac{n!}{n_1! n_2!}. \tag{4.2}$$

From (4.1) and (4.2) one obtains

$$\begin{aligned} \alpha_1^n & \frac{n!}{n_1! n_2!} \left(\frac{n_1(n_2+1)(n_2+2)M_{2,n_2}}{n_2(n_1+1)(n_1+2)M_{2,n_1}} \right)^{\frac{n_2}{2}}, \\ \alpha_2^n & \frac{n!}{n_1! n_2!} \left(\frac{n_2(n_1+1)(n_1+2)M_{2,n_1}}{n_1(n_2+1)(n_2+2)M_{2,n_2}} \right)^{\frac{n_1}{2}}. \end{aligned} \tag{4.3}$$

Thus, for $\alpha_1, \alpha_2 > 0$ such that $\alpha_1 K_1 * \alpha_2 K_2 \in \mathcal{K}_n$ one obtains

$$\begin{aligned} M_{2,n}(\alpha_1 K_1 * \alpha_2 K_2) =& \frac{n}{n_2} \alpha_1^{n_1} \alpha_2^{n_2+2} \frac{n_1!(n_2+2)!}{(n+2)!} M_{2,n_2} \\ =& \frac{n}{(n+1)(n+2)} \left(\frac{n!}{n_1! n_2!}\right)^{\frac{2}{n}} \left(\frac{(n_1+1)(n_1+2)}{n_1} M_{2,n_1}\right)^{\frac{n_1}{n}} \\ & \times \left(\frac{(n_2+1)(n_2+2)}{n_2} M_{2,n_2}\right)^{\frac{n_2}{n}}. \end{aligned}$$

In order to transfer (1.2) from K_1, K_2 to $\alpha_1 K_1 * \alpha_2 K_2$ one therefore has to show

$$\begin{aligned} & \frac{n}{(n+1)(n+2)} \left(\frac{n!}{n_1! n_2!}\right)^{\frac{2}{n}} \left(\frac{n_1!^{\frac{2}{n_1}}}{(n_1+1)^{\frac{1}{n_1}}}\right)^{\frac{n_1}{n}} \left(\frac{n_2!^{\frac{2}{n_2}}}{(n_2+1)^{\frac{1}{n_2}}}\right)^{\frac{n_2}{n}} \\ & \left(= \frac{n}{(n+1)(n+2)} \frac{n!^{2/n}}{((n_1+1)(n_2+1))^{1/n}}\right) \\ & \leq \frac{n}{(n+1)(n+2)} \frac{n!^{2/n}}{(n+1)^{1/n}}. \end{aligned}$$

This, however, is immediate from the inequality

$$(n_1+1)(n_2+1) = n_1 n_2 + n_1 + n_2 + 1 \geq n+1.$$

In order to treat (1.3) we first calculate

$$\begin{aligned} \int_{\alpha_1 K_1 * \alpha_2 K_2} |x''|^4 dx &= \alpha_1^{n_1} M_{0,n_1} \alpha_2^{n_2+4} M_{4,n_2} n_1 \int_0^1 t^{n_1-1}(1-t)^{n_2+4} dt \\ &= \alpha_1^{n_1} \alpha_2^{n_2+4} \frac{n_1!(n_2+4)!}{(n+4)!} M_{4,n_2}, \\ 2\int_{\alpha_1 K_1 * \alpha_2 K_2} |x'|^2 |x''|^2 dx &= 2\alpha_1^{n_1+2} M_{2,n_1} \alpha_2^{n_2+2} M_{2,n_2} \frac{(n_1+2)!(n_2+2)!}{(n+4)!}, \\ \int_{\alpha_1 K_1 * \alpha_2 K_2} |x'|^4 dx & \quad \text{symmetrically.} \end{aligned}$$

A short computation, using (4.2) and (4.3), yields

$$\begin{aligned} \frac{M_{4,n}}{M_{2,n}^2} =& \frac{1}{n^2} \frac{(n+1)(n+2)}{(n+3)(n+4)} \\ & \times \left(\frac{n_1^2(n_1+3)(n_1+4)}{(n_1+1)(n_1+2)} \frac{M_{4,n_1}}{M_{2,n_1}^2} + 2n_1 n_2 + \frac{n_2^2(n_2+3)(n_2+4)}{(n_2+1)(n_2+2)} \frac{M_{4,n_2}}{M_{2,n_2}^2} \right). \end{aligned}$$

Assuming (1.3) for K_1, K_2 one obtains

$$\begin{aligned}\frac{M_{4,n}}{M_{2,n}^2} \leq & \frac{1}{n^2}\frac{(n+1)(n+2)}{(n+3)(n+4)} \\ & \times \left(n_1^2\left(1+\frac{2}{n_1}+\frac{6}{n_1+1}\right)+2n_1n_2+n_2^2\left(1+\frac{2}{n_2}+\frac{6}{n_2+1}\right)\right) \\ = & \frac{(n+1)(n+2)}{(n+3)(n+4)}\left(1+\frac{2}{n}+\frac{6}{n^2}\left(\frac{n_1^2}{n_1+1}+\frac{n_2^2}{n_2+1}\right)\right).\end{aligned}$$

So, in order to conclude (1.3) for $\alpha_1 K_1 * \alpha_2 K_2$ it remains to see

$$\frac{n_1^2}{n_1+1}+\frac{n_2^2}{n_2+1} \leq \frac{n^2}{n+1},$$

which, because of

$$\frac{k^2}{k+1} = k-1+\frac{1}{k+1},$$

is equivalent to

$$\left(\frac{n_2+n_1+2}{(n_1+1)(n_2+1)}=\right)\frac{1}{n_1+1}+\frac{1}{n_2+1} \leq \frac{1}{n+1}+1\left(=\frac{n+2}{n+1}\right).$$

This inequality holds because of $(n_1+1)(n_2+1) \geq n+1$.

Summarizing we have shown that $\mathcal{T}$ is invariant under formation of isotropic normed joins of sets in $\mathcal{T}$. In particular, $\mathcal{T}$ contains all normed cross polytopes X_n.

5. Concluding remarks

Starting from $n=1$, $C := [-\frac{1}{2}, \frac{1}{2}]$ and taking successively cartesian products one obtains the cubes $C^n := [-\frac{1}{2}, \frac{1}{2}]^n$. One easily calculates

$$M_{2,n}(C^n) = \frac{n}{12}, \quad M_{4,n}(C^n) = \frac{n}{5\cdot 16}+\frac{n(n-1)}{144}, \quad \frac{M_{4,n}(C^n)}{M_{2,n}(C^n)^2} = 1+\frac{4}{5n}.$$

We note that for cubes the desired property (ii) of Proposition 1.1 is a consequence of the weak law of large numbers (hint given to the author by H. Vogt), and the corresponding proof for this case was the origin of our estimate (1.1).

Starting from $n=1$, $X_1 := [-1,1]$ and forming successively isotropic normed joins one obtains the normed cross polytopes X_n. One calculates

$$M_{2,n}(X_n) = \frac{n}{2(n+1)(n+2)}n!^{2/n}, \quad \frac{M_{4,n}(X_n)}{M_{2,n}(X_n)^2} = \frac{(n+1)(n+2)}{(n+3)(n+4)}\left(1+\frac{5}{n}\right).$$

Denoting by B_n the n-dimensional Euclidean ball of volume 1 we therefore have the inequalities

$$M_{2,n}(B_n) \leq M_{2,n}(X_n) \leq M_{2,n}(C^n) \leq M_{2,n}(\Delta_n).$$

For $M_{4,n}/M_{2,n}^2$, the chain of inequalities is

$$\frac{M_{4,n}}{M_{2,n}^2}(B_n) \leq \frac{M_{4,n}}{M_{2,n}^2}(C^n) \leq \frac{M_{4,n}}{M_{2,n}^2}(X_n) \leq \frac{M_{4,n}}{M_{2,n}^2}(\Delta_n).$$

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